

Extending Classical Multirate Signal Processing Theory to Graphs – Part II: M-Channel Filter Banks

Oguzhan Teke, *Student Member, IEEE*, P. P. Vaidyanathan, *Fellow, IEEE*

Abstract—This paper builds upon the basic theory of multirate systems for graph signals developed in the companion paper (Part I) and studies M -channel polynomial filter banks on graphs. The behavior of such graph filter banks differs from that of classical filter banks in many ways, the precise details depending on the eigenstructure of the adjacency matrix A . It is shown that graph filter banks represent (linear and) periodically shift-variant systems only when A satisfies the noble identity conditions developed in Part I. It is then shown that perfect reconstruction graph filter banks can always be developed when A satisfies the eigenvector structure satisfied by M -block cyclic graphs and has distinct eigenvalues (further restrictions on eigenvalues being unnecessary for this). If A is actually M -block cyclic then these PR filter banks indeed become practical, i.e., arbitrary filter polynomial orders are possible, and there are robustness advantages. In this case the PR condition is identical to PR in classical filter banks – any classical PR example can be converted to a graph PR filter bank on an M -block cyclic graph. It is shown that for M -block cyclic graphs with all eigenvalues on the unit circle, the frequency responses of filters have meaningful correspondence with classical filter banks. Polyphase representations are then developed for graph filter banks and utilized to develop alternate conditions for alias cancellation and perfect reconstruction, again for graphs with specific eigenstructures. It is then shown that the eigenvector condition on the graph can be relaxed by using similarity transforms.

Index Terms—Multirate processing, graph signals, filter banks on graphs, aliasing on graphs, block-cyclic graphs.

I. INTRODUCTION

In this paper we build upon the basic theory of multirate graph systems developed in the companion paper [1] and study M -channel filter banks on graphs. The M -channel maximally decimated graph filter bank has the form shown in Fig. 1 where $H_k(A)$ and $F_k(A)$ are polynomials in A , and D represents the canonical decimator defined in [1]. As in [1]–[3] the graph adjacency matrix A is assumed to be possibly complex and non symmetric in general. We will see that the behavior of such graph filter banks differs from that of classical filter banks [4], [5] in many ways, the precise details depending on the eigenstructure of the adjacency matrix. An outline of the main results is given below. Needless to say, the problems we address in this paper are inspired by the recent advances reported in [6]–[11]. More can be found in [12]–[15].

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The authors are with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA, 91125, USA. Email: oteke@caltech.edu, ppvnath@systems.caltech.edu.

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A. Scope and outline

In Sec. I-B we review some key equations from the companion paper [1] (renumbered here as (1)–(12)). With the graph adjacency matrix regarded as a shift operator [9], [10], it will be shown in Sec. II-A that the graph filter bank is a shift-variant system, although it is in general not *periodically* shift-variant as in classical time domain filter banks. We then establish the conditions on the adjacency matrix A for the periodically shift-varying property and show that it is exactly identical to the conditions for the existence of graph noble identities (Theorem 2).

Then in Sec. III we consider graphs that satisfy the eigenvector structure of (10) (Ω -graphs). These graphs are more general than M -block cyclic graphs, which satisfy both the eigenvalue and eigenvector conditions of (9) and (10). For such graphs we define band-limited graph signals and polynomial perfect interpolation filters for decimated versions of such signals. This allows us to develop a class of perfect reconstruction filter banks for Ω -graphs (Theorem 4), similar to ideal brickwall filter banks of classical sub-band coding theory. Such graph filter bank designs are usually not practical because the polynomial filters have order $N-1$ (where the graph has N vertices and N can be very large). Furthermore these specific filters for perfect reconstruction are very sensitive to our knowledge of the graph eigenvalues.

In Sec. IV we develop graph filter banks on M -block cyclic graphs (defined and studied in Sec. V of [1]). For such graphs the eigenvalues and eigenvectors are *both* constrained as in (9), (10). We show that for such graphs the condition for perfect reconstruction is very similar to the PR condition in classical filter banks (Theorem 6). For such graphs, it is therefore possible to design PR filter banks by starting from any classical PR system. In particular it is possible to obtain PR systems with arbitrarily small orders (independent of the size of the graph) for the polynomial filters. Furthermore the PR solutions $\{H_k(A), F_k(A)\}$ are not sensitive to graph eigenvalues.

In Sec. V we consider polyphase representations for graph filter banks. This is useful to obtain alternative theoretical representations, as well as in implementation. Unlike classical filter banks, these polyphase representations are not always valid. They are valid only for those graphs that satisfy the noble identity requirements (Theorem 3 of [1]). For such graphs, the PR condition and the alias cancellation condition can further be expressed in terms of the analysis and synthesis polyphase matrices if the graphs also satisfy the M -block cyclic property (Theorems 10 and 11). Interestingly these alias cancellation conditions are somewhat similar to the pseudo-

circulant property developed for classical filter banks in [4].

In Sec. VI we consider frequency responses of graph filter banks inspired by similar ideas in [8], [10]. We will see that this concept can be meaningfully developed for filter banks on M -cyclic graphs with all eigenvalues on the unit circle, but not for arbitrary graphs.

Finally in Sec. VII we show that the eigenvector structure in (10) (Ω -structure) can be relaxed simply by considering a transformed graph based on similarity transformations. This generalization therefore extends many of the results in this and the companion paper [1] to more general graphs. In short, all results that we developed for Ω -graphs (e.g., Theorems 3 and 4) generalize to arbitrary graphs. Similarly all results which we developed for M -block cyclic graphs (e.g., Theorems 5 and 6) generalize to graphs that are subject only to the eigenvalue constraint (9) and not the eigenvector constraint (10).

Section IX concludes the paper. This paper follows the notation described in the companion paper [1].

B. Review from [1]

We defined M -fold graph decimation operator by the matrix

$$\mathbf{D} = \begin{bmatrix} \mathbf{I}_{N/M} & \mathbf{0}_{N/M} & \cdots & \mathbf{0}_{N/M} \end{bmatrix} \in \mathcal{C}^{(N/M) \times N}. \quad (1)$$

We then showed that the following noble identities

$$\mathbf{D} H(\mathbf{A}^M) = H(\bar{\mathbf{A}}) \mathbf{D}, \quad (2)$$

$$H(\mathbf{A}^M) \mathbf{D}^T = \mathbf{D}^T H(\bar{\mathbf{A}}), \quad (3)$$

are simultaneously satisfied for all polynomial filters $H(\cdot)$ if and only if the following two equations are satisfied: \mathbf{A} being the adjacency matrix of the graph, \mathbf{A}^M has the form

$$\mathbf{A}^M = \begin{bmatrix} (\mathbf{A}^M)_{1,1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{A}^M)_{2,2} \end{bmatrix}, \quad (4)$$

and

$$\bar{\mathbf{A}} = \mathbf{D} \mathbf{A}^M \mathbf{D}^T \in \mathcal{M}^{N/M}, \quad (5)$$

where $(\mathbf{A}^M)_{1,1} \in \mathcal{M}^{N/M}$. We noted that the condition in (4) is not trivial in the sense that even very simple graphs can fail to satisfy it. We also showed that M -block cyclic graphs satisfy the noble identity condition (4). We called a graph (balanced) M -block cyclic when it had the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_M \\ \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{M-1} & \mathbf{0} \end{bmatrix} \in \mathcal{M}^N, \quad (6)$$

where each $\mathbf{A}_j \in \mathcal{M}^{N/M}$. For the eigenvalue decomposition of the adjacency matrix $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$, when we have the following double indexing scheme

$$\mathbf{\Lambda} = \text{diag}([\lambda_{1,1} \cdots \lambda_{1,M} \cdots \lambda_{N/M,1} \cdots \lambda_{N/M,M}]), \quad (7)$$

$$\mathbf{V} = [\mathbf{v}_{1,1} \cdots \mathbf{v}_{1,M} \cdots \mathbf{v}_{N/M,1} \cdots \mathbf{v}_{N/M,M}], \quad (8)$$

we showed that eigenvalues and eigenvectors of M -block cyclic graphs have the following relation

$$\lambda_{i,j+k} = w^k \lambda_{i,j}, \quad (9)$$

$$\mathbf{v}_{i,j+k} = \mathbf{\Omega}^k \mathbf{v}_{i,j}, \quad (10)$$

where

$$w = e^{-j2\pi/M}, \quad (11)$$

$$\mathbf{\Omega} = \text{diag}([1 \ w^{-1} \ w^{-2} \ \cdots \ w^{-(M-1)}]) \otimes \mathbf{I}_{N/M}. \quad (12)$$

In the rest of the paper the eigenvector structure in (10) will be referred to as the Ω -structure. A graph with eigenvectors satisfying the condition in (10) will be referred to as an Ω -graph. Notice that an Ω -graph can have arbitrary eigenvalues. An M -block cyclic graph is also an Ω -graph since its eigenvectors have the Ω -structure. Furthermore, any circulant graph is an Ω -graph since columns of the properly permuted DFT matrix have the structure in (10).

II. GRAPH FILTER BANKS

In the companion paper [1], we found that multirate building blocks defined on graphs satisfy identities similar to classical multirate identities only under certain conditions on the adjacency matrix \mathbf{A} . In the case of filter banks, even the simplest maximally decimated filter bank (the lazy filter bank of Fig. 3 of [1]) may or may not satisfy perfect reconstruction (unlike in the classical case). These were elaborated in Sec. II of [1].

In this section we consider filter banks in greater detail. Fig. 1 shows a maximally decimated graph filter bank where the analysis filters $H_k(\mathbf{A})$ and the synthesis filters $F_k(\mathbf{A})$ are polynomials in \mathbf{A} , and \mathbf{D} is as in (1). In the classical case, a maximally decimated filter bank is known to be a periodically time-varying system unless aliasing is completely canceled, in which case it becomes a time-invariant system. It is possible to get a somewhat analogous property for filter banks on graphs, but only under some conditions. In this section we first develop these results. We then study a class of filter banks that are analogous to ideal brickwall filters (bandlimited filter banks in the classical case) and show that perfect reconstruction can be achieved under some constraints on the eigen-structure of the graph \mathbf{A} .

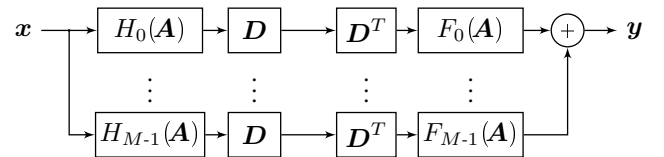


Fig. 1. An M -channel maximally decimated filter bank on a graph with adjacency matrix \mathbf{A} . Here $H_k(\mathbf{A})$ and $F_k(\mathbf{A})$ are polynomials in \mathbf{A} (so they are linear shift-invariant systems [1]). The decimation matrix \mathbf{D} is as in (1) with decimation ratio M . Overall response of the filter bank is denoted as $T(\mathbf{A})$, that is, $\mathbf{y} = T(\mathbf{A}) \mathbf{x}$.

A. Graph Filter Banks as Periodically Shift-Varying Systems

In classical filter banks where the filters are polynomials in the shift operator z^{-1} , the maximally decimated analysis/synthesis system is a linear periodically shift-variant

(LPSV(M)) system (i.e., the system from \mathbf{x} to \mathbf{y} in Fig. 1 is LPSV). This is always true regardless of what the filter coefficients are. In particular, if the filter coefficients are such that this LPSV(M) system reduces to an LTI system, then the system can be shown to be alias-free (and vice versa) [4]. Furthermore if this LTI system is a pure delay cz^{-n_0} then the system has the perfect reconstruction (PR) property. Since our main goal is to get insights into a parallel theory for graph filter banks, we now discuss the role of the periodically shift-varying property for graph filter banks with polynomial filters.

For the filter bank in Fig. 1, let $T_k(\mathbf{A})$ denote the response of the k^{th} channel. Therefore we have,

$$T_k(\mathbf{A}) = F_k(\mathbf{A}) \mathbf{D}^T \mathbf{D} H_k(\mathbf{A}), \quad (13)$$

and the overall response from \mathbf{x} to \mathbf{y} is

$$T(\mathbf{A}) = \sum_{k=0}^{M-1} T_k(\mathbf{A}) = \sum_{k=0}^{M-1} F_k(\mathbf{A}) \mathbf{D}^T \mathbf{D} H_k(\mathbf{A}). \quad (14)$$

Notice that $T_k(\mathbf{A})$ is a linear operator. However, the DU operator, $\mathbf{D}^T \mathbf{D}$, does not commute with \mathbf{A} in general, hence response of the k^{th} channel is shift-varying. That is to say,

$$T_k(\mathbf{A}) \mathbf{A} \neq \mathbf{A} T_k(\mathbf{A}), \quad (15)$$

for arbitrary analysis and synthesis filters. Therefore we have proved:

Theorem 1 (Graph filter banks are shift-varying). *An arbitrary maximally decimated M -channel filter bank on an arbitrary graph is in general a linear but shift-varying system (i.e., the mapping from \mathbf{x} to \mathbf{y} in Fig. 1 is linear and shift-varying). \diamond*

This is similar to classical multirate theory where an arbitrary filter bank is a time-varying system [4]. In fact, filter banks in the classical theory are periodically time-varying systems with period M . This leads to the question: is the filter bank on a graph periodically shift-varying?

In classical theory, when a system relates the input $x(n)$ to the output $y(n)$ in the following way $y(n) = \sum_k a_n(k) x(n-k)$, a linear and periodically time-varying system with period M is defined by the defining equation $a_{n+M}(k) = a_n(k)$ in [4]. It can be shown that a linear system satisfies this relation if and only if the following is true: if $x(n)$ produces output $y(n)$, then $x(n+M)$ produces output $y(n+M)$. Motivated by this, we present the following definition.

Definition 1 (Periodically shift-varying system). *A linear system \mathbf{H} on a graph \mathbf{A} is said to be periodically shift-varying with period M , LPSV(M), if $\mathbf{A}^M \mathbf{H} = \mathbf{H} \mathbf{A}^M$. This reduces to shift-invariance when $M = 1$. \diamond*

Using this definition, we state the following result for the periodically shift-varying response of a maximally decimated M -channel filter bank on graphs.

Theorem 2 (Periodically shift-varying filter banks). *The graph FB in Fig. 1 is LPSV(M) for all choices of the polynomial filters $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$, if and only if the adjacency matrix of the graph satisfies the noble identity condition in (4). \diamond*

Proof: The LPSV(M) property, by Definition 1, is equivalent to

$$\sum_{k=0}^{M-1} T_k(\mathbf{A}) \mathbf{A}^M = \mathbf{A}^M \sum_{k=0}^{M-1} T_k(\mathbf{A}), \quad (16)$$

Since $T_k(\mathbf{A})$ is as in (13), this is true for all polynomial filters $H_k(\mathbf{A})$ and $F_k(\mathbf{A})$ if and only if

$$\mathbf{D}^T \mathbf{D} \mathbf{A}^M = \mathbf{A}^M \mathbf{D}^T \mathbf{D}. \quad (17)$$

Partition \mathbf{A}^M as in Eq. (18) of [1], and substitute into (17). Then the result is $(\mathbf{A}^M)_{1,2} = \mathbf{0}$ and $(\mathbf{A}^M)_{2,1} = \mathbf{0}$, which is the same as the condition (4). Conversely, if (4) holds it is obvious that (17) holds (because $\mathbf{D}^T \mathbf{D}$ is as in Eq. (13) of [1]), so the LPSV(M) property (16) follows. \blacksquare

III. GRAPH FILTER BANKS ON Ω -GRAPHS

In this section we will construct maximally decimated M -channel filter banks with perfect reconstruction (PR) property on Ω -graphs. These filter banks are ideal in the sense that each channel has a particular sub-band and there is no overlap between different channels. The key point here is that this construction uses analysis and synthesis filters that are *both* alias-free (diagonal matrices in the frequency domain according to Def. 7 of [1]). If the filters do not have to satisfy this restriction, then the problem of constructing PR filter banks becomes rather trivial. (See Sec. III of [1].) In fact, one can find low order polynomial filters by first designing unconstrained filters (or, polynomial filters with high degree as in [6], [12]) with PR property, then computing their low order polynomial approximations. This approach results in a filter bank with approximate PR property, with a trade-off between the approximation error and the degree of the filters. However, our approach here is to directly design alias-free filters. Later in Sec. IV we will study how we can design low order polynomial filters directly.

Remember that Ω -graphs do not have any constraints on eigenvalues, however, the eigenvectors of Ω -graphs satisfy the Ω -structure given in (10). In Sec. VII, we will show how this constraint on eigenvectors can be removed by generalizing the definition of the decimator \mathbf{D} .

Remember from Eq. (68) of [1] that DU operation, $\mathbf{D}^T \mathbf{D}$, results in aliasing for an arbitrary graph signal. Nonetheless, we can still recover the input signal from the output of $\mathbf{D}^T \mathbf{D}$ if the input signal has zeros in its graph Fourier transform. To discuss this further, we define *band-limited* signals on graphs as follows:

Definition 2 (Band-limited graph signals on Ω -graphs). *Let \mathbf{A} be the adjacency matrix of an Ω -graph with the following eigenvalue decomposition $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$. A signal \mathbf{x} on this Ω -graph said to be k^{th} -band-limited when its graph Fourier transform, $\hat{\mathbf{x}} = \mathbf{V}^{-1} \mathbf{x}$, has zeros in the following way:*

$$\hat{x}_{i,j} = 0, \quad 1 \leq j \leq M, \quad j \neq k, \quad 1 \leq i \leq N/M, \quad (18)$$

where we used the double indexing scheme similar to (7) and (8) to denote the graph Fourier coefficients $\hat{x}_{i,j}$. \diamond

In the literature, there are different notions and definitions for band-limited graph signals [16]–[20]. Under an appropriate

re-indexing of the eigenvalues (and the eigenvectors), our notion of k^{th} -band-limited signal is similar to the one in [16] with bandwidth N/M . This is consistent with our purpose of constructing M -channel graph filter banks.

When a graph signal is k^{th} -band-limited according to Definition 2, due to Eq. (68) of [1], the output spectrum of DU operation becomes as follows.

$$\hat{y}_{i,1} = \hat{y}_{i,2} = \dots = \hat{y}_{i,M} = \frac{1}{M} \hat{x}_{i,k}. \quad (19)$$

In this case we can recover the input signal from the output of the DU operator since only one of the aliasing frequency components is non-zero. For this purpose, let \mathbf{F} be a linear filter with frequency response $\mathbf{V}^{-1} \mathbf{F} \mathbf{V} = \mathbf{I}_{N/M} \otimes M \mathbf{e}_k \mathbf{e}_k^T$, where \mathbf{e}_k is the k^{th} element of the standard basis for \mathcal{C}^M . Then, consider the following system,

$$\mathbf{z} = \mathbf{F} \mathbf{D}^T \mathbf{D} \mathbf{x}. \quad (20)$$

Due to Eq. (66) of [1] and the construction of \mathbf{F} above, in the graph Fourier domain (20) translates to the following,

$$\begin{aligned} \hat{\mathbf{z}} &= \left(\mathbf{I}_{N/M} \otimes M \mathbf{e}_k \mathbf{e}_k^T \right) \left(\mathbf{I}_{N/M} \otimes \frac{1}{M} \mathbf{1} \mathbf{1}^T \right) \hat{\mathbf{x}} \\ &= \left(\mathbf{I}_{N/M} \otimes \mathbf{e}_k \mathbf{1}^T \right) \hat{\mathbf{x}}. \end{aligned} \quad (21)$$

Since $\hat{\mathbf{x}}$ is assumed to be k^{th} -band-limited according to Definition 2, we get $\hat{\mathbf{z}} = \hat{\mathbf{x}}$. That is, we can reconstruct the original signal from its decimated version using the linear reconstruction filter \mathbf{F} .

Notice that the frequency response of \mathbf{F} is a diagonal matrix by its definition. Therefore \mathbf{F} is an alias-free filter due to Definition 7 of [1]. Furthermore, when the graph is assumed to have distinct eigenvalues, \mathbf{F} can be realized as a polynomial filter due to Theorem 11 of [1]. We have therefore proved the following theorem.

Theorem 3 (Polynomial interpolation filters for Ω -graphs). *Let \mathbf{A} be the adjacency matrix of an Ω -graph. Let \mathbf{x} be a k^{th} -band-limited signal on the graph. Then, there exists an interpolation filter \mathbf{F} that recovers \mathbf{x} from $\mathbf{D}\mathbf{x}$. When the eigenvalues are distinct, \mathbf{F} can be a polynomial in \mathbf{A} , that is, $\mathbf{F} = \mathbf{F}(\mathbf{A})$.* \diamond

The recovery of missing samples in graph signals is also discussed in various settings [16], [21], [22].

In the following, we will discuss how we can write an arbitrary full-band signal as a sum of k^{th} -band-limited signals. For this purpose, consider the following identity,

$$\mathbf{I}_N = \sum_{k=1}^M \mathbf{I}_{N/M} \otimes \mathbf{e}_k \mathbf{e}_k^T. \quad (22)$$

Then we can write $\hat{\mathbf{x}} = \sum_{k=1}^M \hat{\mathbf{x}}_k$, where

$$\hat{\mathbf{x}}_k = \left(\mathbf{I}_{N/M} \otimes \mathbf{e}_k \mathbf{e}_k^T \right) \hat{\mathbf{x}}. \quad (23)$$

With the construction in (23), $\hat{\mathbf{x}}_k$ is a k^{th} -band-limited signal. Notice that (23) is a frequency domain relation. In the graph signal domain, consider a linear filter \mathbf{H}_{k-1} with frequency response

$$\widehat{\mathbf{H}}_{k-1} = \mathbf{V}^{-1} \mathbf{H}_{k-1} \mathbf{V} = \mathbf{I}_{N/M} \otimes \mathbf{e}_k \mathbf{e}_k^T. \quad (24)$$

Then, we have $\mathbf{x} = \sum_{k=1}^M \mathbf{x}_k$, where $\mathbf{x}_k = \mathbf{H}_{k-1} \mathbf{x}$. Since \mathbf{x}_k is a k^{th} -band-limited graph signal, we can reconstruct it from its decimated version using the interpolation filter discussed in Theorem 3. For this purpose, let \mathbf{F}_{k-1} be a linear filter with frequency response $\mathbf{V}^{-1} \mathbf{F}_{k-1} \mathbf{V} = \mathbf{I}_{N/M} \otimes M \mathbf{e}_k \mathbf{e}_k^T$. We will then have:

$$\mathbf{x}_k = \mathbf{F}_{k-1} \mathbf{D}^T \mathbf{D} \mathbf{x}_k = \mathbf{F}_{k-1} \mathbf{D}^T \mathbf{D} \mathbf{H}_{k-1} \mathbf{x}. \quad (25)$$

So we have proved the following theorem.

Theorem 4 (PR filter banks on Ω -graphs). *Let \mathbf{A} be the adjacency matrix of an Ω -graph with the following eigenvalue decomposition $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$. Now consider the maximally decimated filter bank of Fig. 1 with analysis and synthesis filters as follows:*

$$\mathbf{H}_{k-1} = \mathbf{V} \left(\mathbf{I}_{N/M} \otimes \mathbf{e}_k \mathbf{e}_k^T \right) \mathbf{V}^{-1}, \quad \mathbf{F}_{k-1} = M \mathbf{H}_{k-1} \quad (26)$$

for $1 \leq k \leq M$. This is a perfect reconstruction system, that is, $\mathbf{T}(\mathbf{A}) \mathbf{x} = \mathbf{x}$ for all graph signals \mathbf{x} . When the eigenvalues are distinct, \mathbf{H}_k 's can be designed to be polynomials in \mathbf{A} , that is, $\mathbf{H}_{k-1} = \mathbf{H}_{k-1}(\mathbf{A})$. \diamond

Notice that the frequency response of each filter, (26), has N/M nonzero values. These are similar to ‘‘ideal’’ band-limited filters (with bandwidth $2\pi/M$) in classical theory. In classical filter bank theory it is well known that an M -channel maximally decimated filter bank has perfect reconstruction if the filters are ideal ‘‘brickwall’’ filters chosen as

$$H_k(e^{j\omega}) = \begin{cases} 1, & 2\pi k/M \leq \omega \leq 2\pi(k+1)/M, \\ 0, & \text{otherwise,} \end{cases} \quad (27)$$

and $F_k(e^{j\omega}) = M H_k(e^{j\omega})$. The result of Theorem 4 for graph filter banks is analogous to that classical result. Notice that in classical theory, ideal filters have infinite duration impulse responses, whereas the graph filters are polynomials in \mathbf{A} with at most N taps.

Figure 2(a) shows the details of one channel of the brickwall analysis bank, and Fig. 2(b) shows the corresponding channel of the synthesis bank. The analysis filters have the form $\mathbf{H}_k = \mathbf{V} \mathbf{S}_k \mathbf{V}^{-1}$ where \mathbf{V}^{-1} is the graph Fourier transform matrix and $\mathbf{S}_k = \mathbf{I}_{N/M} \otimes \mathbf{e}_k \mathbf{e}_k^T$ is a diagonal matrix (band selector) which retains N/M outputs of \mathbf{V}^{-1} and sets the rest to zero. For example if $N=6$ and $M=3$ the three selector matrices are

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{S}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (28)$$

Once the appropriate outputs of the k^{th} band have been selected, the matrix \mathbf{V} in \mathbf{H}_k is used to convert the subband signal back to the ‘‘graph vertex domain.’’ (This is similar to implementing the filtering operation in the frequency domain and taking inverse Fourier transform to come back to time domain.) This graph domain subband signal is then decimated by \mathbf{D} . For reconstruction, the synthesis filter \mathbf{F}_k is similarly

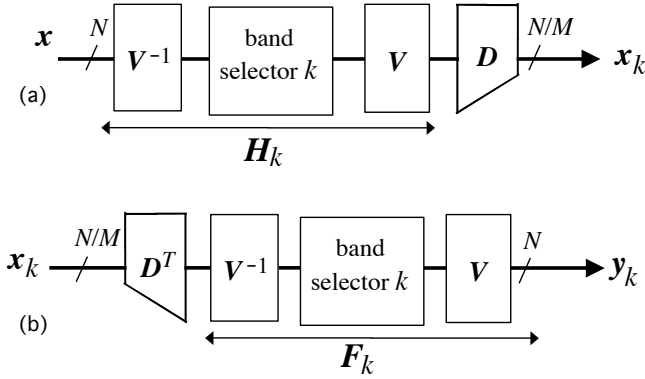


Fig. 2. (a) The k^{th} channel of a graph filter bank, and (b) details of this channel when the filters are brickwall filters as defined in Theorem 4.

used. Thus, the implementation of the brick wall filter bank is not merely a matter of using the graph Fourier operator V^{-1} , it also involves band selection, inverse transformation, and decimation. Since V^{-1} is common to all analysis filters, it contributes to a complexity of N^2 multiplications. The band selector S_k , and the matrices V and D can be combined and implemented with $(N/M)^2$ multiplications for each k , so there is a total of N^2/M multiplications for this part. So the decimated analysis bank has complexity of about $N^2 + N^2/M$ which is $O(N^2)$. Once again, by approximating these brickwall filters with polynomial filters with order L we can reduce the complexity to NL . (See Fig. 7 of [1].)

The perfect reconstruction result in Theorem 4 is restricted to Ω -graphs. However this restriction can be removed by applying a similarity transformation to the graph as described later in Sec. VII.

As a final remark, filters in (26) are not unique in the sense that we can design different filters and still have $T(\mathbf{A}) = \mathbf{I}$ in the filter bank.

IV. GRAPH FILTER BANKS ON M -BLOCK CYCLIC GRAPHS

In Theorem 4 aliasing was totally suppressed in each channel by the use of ideal filters (26), which explains why the filter polynomials had order $N-1$. But if we resort to cancellation of aliasing among different channels, it is less restrictive on the filters. While this idea is not easy to develop for arbitrary graphs, the theory can be developed under some further assumptions on the graph, namely that \mathbf{A} be M -block cyclic as we shall see. We will see that such filter banks have many advantages compared to those given by Theorem 4, which does not use the M -block cyclic assumption and applies to more general class of Ω -graphs.

As shown in Theorem 4 of [1], M -block cyclic graphs satisfy the noble identity condition in (4). Therefore, Theorem 2 shows that a maximally decimated M -channel filter bank on an M -block cyclic graph is a periodically shift-varying system with period M . This can be interpreted as aliasing when the graph \mathbf{A} is diagonalizable (Theorem 10 and Fig. 10 of [1]). As in the classical case, it is possible to cancel out aliasing components arising from different channels, and in fact, achieve perfect reconstruction, that is, $T(\mathbf{A}) = \mathbf{A}^n$ for

some integer n as we shall see.¹ We begin by proving the following result:

Theorem 5 (PR filter banks on M -block cyclic graphs). *Consider the graph filter bank of Fig. 1 and assume that the adjacency matrix of the graph is diagonalizable M -block cyclic. With no further restrictions on the graph, the system has perfect reconstruction if and only if*

$$\sum_{k=0}^{M-1} F_k(\lambda) H_k(w^l \lambda) = M \lambda^n \delta(l), \quad (29)$$

for some n , for all $\lambda \in \mathcal{C}$, and for all l in $0 \leq l \leq M-1$, where $\delta(\cdot)$ is the discrete Dirac function. \diamond

Proof: The overall response of the system in Fig. 1 can be written as

$$\begin{aligned} T(\mathbf{A}) &= \sum_{k=0}^{M-1} F_k(\mathbf{A}) D^T D H_k(\mathbf{A}), \\ &= \frac{1}{M} \sum_{l=0}^{M-1} \sum_{k=0}^{M-1} F_k(\mathbf{A}) \Omega^l H_k(\mathbf{A}), \end{aligned} \quad (30)$$

where we have used Eq. (58) of [1]. For simplicity, define:

$$S_l(\mathbf{A}) = \sum_{k=0}^{M-1} F_k(\mathbf{A}) \Omega^l H_k(\mathbf{A}). \quad (31)$$

Therefore, the response of the system is:

$$T(\mathbf{A}) = \frac{1}{M} \left(\underbrace{S_0(\mathbf{A})}_{\text{Polynomial}} + \underbrace{S_1(\mathbf{A}) + \dots + S_{M-1}(\mathbf{A})}_{\text{Alias components}} \right). \quad (32)$$

Notice that $S_0(\mathbf{A})$ is the sum of products of polynomials in \mathbf{A} , therefore it is also a polynomial in \mathbf{A} , hence it is alias-free since \mathbf{A} is assumed to be diagonalizable in [1]. However, for $l \geq 1$, there exists Ω^l term in each $S_l(\mathbf{A})$ in (31), which does not commute with \mathbf{A} in general. As a result $S_l(\mathbf{A})$ is not shift-invariant and results in aliasing (Theorem 10 of [1]). Perfect reconstruction $T(\mathbf{A}) = \mathbf{A}^n$ can be achieved by imposing:

$$S_0(\mathbf{A}) = M \mathbf{A}^n, \quad \sum_{l=1}^{M-1} S_l(\mathbf{A}) = \mathbf{0}. \quad (33)$$

The second equation above is the alias cancellation condition. Using the eigenvalue decomposition of the adjacency matrix, $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$, the first condition in (33) reduces to:

$$\mathbf{V} \left(\sum_{k=0}^{M-1} F_k(\mathbf{\Lambda}) H_k(\mathbf{\Lambda}) \right) \mathbf{V}^{-1} = M \mathbf{V} \mathbf{\Lambda}^n \mathbf{V}^{-1}. \quad (34)$$

Since $\mathbf{\Lambda}$ is a diagonal matrix consisting of eigenvalues, this can be further simplified to

$$\sum_{k=0}^{M-1} F_k(\lambda_{i,j}) H_k(\lambda_{i,j}) = M \lambda_{i,j}^n, \quad (35)$$

for all eigenvalues, $\lambda_{i,j}$, of the adjacency matrix \mathbf{A} .

¹In analogy with classical filter banks where $T(z) = z^{-n}$ signifies the PR property, we take $T(\mathbf{A}) = \mathbf{A}^n$ to be the PR property. But this makes practical sense only in situations where \mathbf{A} is invertible so that the distortion \mathbf{A}^n can be canceled.

Now consider the second condition in (33). With the eigenvalue decomposition, it reduces to

$$\sum_{l=1}^{M-1} \sum_{k=0}^{M-1} F_k(\Lambda) \mathbf{V}^{-1} \Omega^l \mathbf{V} H_k(\Lambda) = \mathbf{0}. \quad (36)$$

Since \mathbf{A} is M -block cyclic, it satisfies the eigenvector structure in (10). So we can use Eq. (63) of [1] and re-write (36) as:

$$\sum_{l=1}^{M-1} \sum_{k=0}^{M-1} F_k(\Lambda) \Pi_l H_k(\Lambda) = \mathbf{0}. \quad (37)$$

Notice that the permutation matrix Π_l is defined as the l^{th} power of the cyclic matrix of size M (Eq. (64) of [1]). Since the supports of $\mathbf{C}_M^{l_1}$ and $\mathbf{C}_M^{l_2}$ have no common index for different l_1 and l_2 , supports of Π_{l_1} and Π_{l_2} will also have no common index. Furthermore, due to Λ being a diagonal matrix, the support of the term $F_k(\Lambda) \Pi_l H_k(\Lambda)$ will be the same as the support of Π_l . Combining both arguments, we can say that (37) holds if and only if the inner sum is zero for each l , that is,

$$\sum_{k=0}^{M-1} F_k(\Lambda) \Pi_l H_k(\Lambda) = \mathbf{0}, \quad \forall l \in \{1, \dots, M-1\}. \quad (38)$$

Since Λ is a diagonal matrix with the ordering scheme in (7), the permutation Π_l circularly shifts each eigenvalue of an eigenfamily. Hence, (38) is equivalent to:

$$\sum_{k=0}^{M-1} F_k(\lambda_{i,j+l}) H_k(\lambda_{i,j}) = 0, \quad (39)$$

for all $1 \leq l \leq M-1$. Since \mathbf{A} is M -block cyclic, eigenvalues of \mathbf{A} satisfy (9). Then (39) can be written as:

$$\sum_{k=0}^{M-1} F_k(\lambda_{i,j+l}) H_k(w^{M-l} \lambda_{i,j+l}) = 0. \quad (40)$$

By changing the index variables, (40) can be simplified as follows:

$$\sum_{k=0}^{M-1} F_k(\lambda_{i,j}) H_k(w^l \lambda_{i,j}) = 0, \quad (41)$$

for all $1 \leq l \leq M-1$ and for all eigenvalues, $\lambda_{i,j}$, of the adjacency matrix \mathbf{A} . Combining (35) and (41), if the filter bank in Fig. 1 provides PR, (29) should be satisfied for all eigenvalues of \mathbf{A} . Since we want PR independent of the graph, (29) should be satisfied for all $\lambda \in \mathcal{C}$. Hence, it is a necessary condition.

Conversely, assume that the filters satisfy (29), hence (35) and (41) are satisfied. Since the graph is assumed to be diagonalizable M -block cyclic, (35) and (41) are equivalent to (33). Therefore, the overall response of the filter bank is \mathbf{A}^n , where n satisfies (35). So the filter bank has the PR property. ■

It is quite interesting to observe that the condition (29) on the graph filters is the same as the PR condition in classical multirate theory. We state this equivalence in the following theorem.

Theorem 6 (PR condition equivalence). *A set of polynomials, $\{H_k(\lambda), F_k(\lambda)\}$, provides PR in maximally decimated M -channel FB on all M -block cyclic graphs with diagonalizable adjacency matrix if and only if $\{H_k(z), F_k(z)\}$ provides PR in the classical maximally decimated M -channel FB. ◇*

Proof: The PR condition (29) is the same as the PR condition for the classical maximally decimated M -channel filter bank [4]. ■

At the moment of this writing, we do not have examples of graphs other than M -block cyclic for which such results can be developed. At the expense of restraining the eigenvalues of the adjacency matrix to have the structure in (9), Theorem 5 offers three significant benefits compared to construction of PR filter banks on Ω -graphs discussed in Sec. III.

First of all, (29) puts a condition on the filter coefficients independent of the graph as long as the graph is M -block cyclic with diagonalizable adjacency matrix. A change in the adjacency matrix does not affect the PR property. As a result, the response of the overall graph filter bank is robust to ambiguities in the adjacency matrix.

Secondly, eigenvalues of the adjacency matrix \mathbf{A} do not need to be distinct since the condition solely depends on the filter coefficients.

Lastly, filter banks on M -block cyclic graphs are legitimate generalization of the classical multirate theory to graph signals due to Theorem 6. In order to design PR filter banks on an M -block cyclic graph, we can use any algorithm developed in the classical multirate theory [4], [23]. As an example, consider the following set of polynomials,

$$\begin{aligned} H_0(\lambda) &= 5+2\lambda+\lambda^3+2\lambda^4+\lambda^5, & F_0(\lambda) &= 3\lambda^3-2\lambda^4+\lambda^5, \\ H_1(\lambda) &= 2+\lambda+2\lambda^3+4\lambda^4+2\lambda^5, & F_1(\lambda) &= -8\lambda^3+5\lambda^4-2\lambda^5, \\ H_2(\lambda) &= \lambda^3+2\lambda^4+\lambda^5, & F_2(\lambda) &= 1+13\lambda^3-8\lambda^4+3\lambda^5. \end{aligned} \quad (42)$$

In classical theory, this is a 3-channel PR filter bank [4]. Notice that these polynomials satisfy (29) with $n = 5$. Therefore, overall response of the filter bank constructed with the polynomials in (42) on any 3-block cyclic graph will be $T(\mathbf{A}) = \mathbf{A}^5$.

For a randomly generated adjacency matrix of a 3-block cyclic graph, channel responses of the filter bank utilizing the filters in (42) are shown in Fig. 3. Response of each channel has non-zero blocks with large values in it (relative to the overall response of the FB), which results in large alias terms. Notice that some blocks of the channel responses cancel out each other exactly so that overall response, $T(\mathbf{A})$, is equal to \mathbf{A}^5 (up to numerical precision) as seen from Fig. 3(f).

V. POLYPHASE REPRESENTATIONS

In [1], for any given polynomial filter $H(\mathbf{A})$ on any graph \mathbf{A} , we wrote its Type-1 polyphase representation as

$$H(\mathbf{A}) = \sum_{l=0}^{M-1} \mathbf{A}^l E_l(\mathbf{A}^M), \quad (43)$$

and its Type-2 polyphase decomposition as

$$H(\mathbf{A}) = \sum_{l=0}^{M-1} \mathbf{A}^{M-1-l} R_l(\mathbf{A}^M), \quad (44)$$

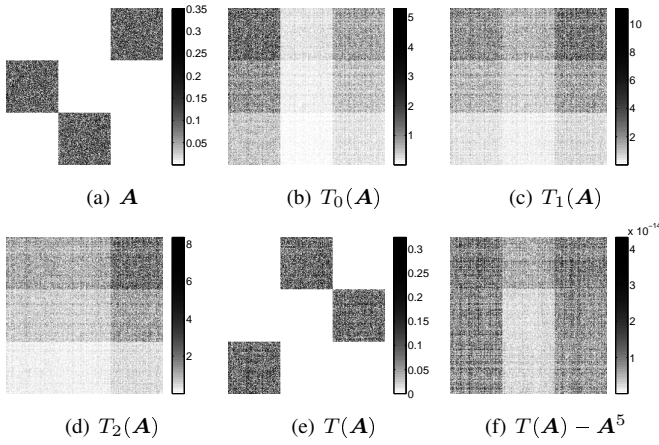


Fig. 3. (a) Adjacency matrix of a 3-block cyclic graph of size 210 with randomly generated complex edge weights. Response of (b) the zeroth channel, (c) the first channel, (d) the second channel of 3-channel FB in Fig. 1 on graph given in (a). (e) is the overall response of the FB. (f) is the difference between $T(\mathbf{A})$ and \mathbf{A}^5 . All figures show the element-wise absolute values of the matrices.

where $E_k(\cdot)$'s and $R_k(\cdot)$'s are polynomials.

For the polyphase implementation of the filter bank in Fig. 1, we decompose the analysis filters using Type-1 polyphase structure in (43) and synthesis filters with Type-2 decomposition in (44). Then we have,

$$H_k(\mathbf{A}) = \sum_{l=0}^{M-1} \mathbf{A}^l E_{k,l}(\mathbf{A}^M), \quad F_k(\mathbf{A}) = \sum_{l=0}^{M-1} \mathbf{A}^{M-1-l} R_{l,k}(\mathbf{A}^M). \quad (45)$$

Using the polyphase implementation of decimation and interpolation filters given in Eq. (34) and Eq. (35) of [1] (they are described schematically in Fig. 4 and Fig. 5 of [1], respectively), we can define the polyphase component matrices as follows:

$$(\mathbf{E}(\bar{\mathbf{A}}))_{i,j} = E_{i-1,j-1}(\bar{\mathbf{A}}), \quad (\mathbf{R}(\bar{\mathbf{A}}))_{i,j} = R_{i-1,j-1}(\bar{\mathbf{A}}), \quad (46)$$

for $1 \leq i, j \leq M$ where $\mathbf{E}(\bar{\mathbf{A}}) \in \mathcal{M}^N$ is the polyphase matrix for the analysis filters, $\mathbf{R}(\bar{\mathbf{A}}) \in \mathcal{M}^N$ is the polyphase matrix for the synthesis filters, and $\bar{\mathbf{A}}$ is the adjusted shift operator given in (5). Notice that each block $(\mathbf{E}(\bar{\mathbf{A}}))_{i,j}$ and $(\mathbf{R}(\bar{\mathbf{A}}))_{i,j}$ is a polynomial in $\bar{\mathbf{A}}$, whereas overall polyphase matrices $\mathbf{E}(\bar{\mathbf{A}})$ and $\mathbf{R}(\bar{\mathbf{A}})$ are not polynomials in $\bar{\mathbf{A}}$.

Using the polyphase matrices defined in (46), we can implement the filter bank in Fig. 1 as in Fig. 4(a). This can be redrawn as in Fig. 4(b) where $\mathbf{P}(\bar{\mathbf{A}}) = \mathbf{R}(\bar{\mathbf{A}}) \mathbf{E}(\bar{\mathbf{A}})$. Due to partitioning of the polyphase component matrices in (46), we have the following result for the partitions of $\mathbf{P}(\bar{\mathbf{A}})$:

$$(\mathbf{P}(\bar{\mathbf{A}}))_{i,j} = \sum_{k=0}^{M-1} R_{i-1,k}(\bar{\mathbf{A}}) E_{k,j-1}(\bar{\mathbf{A}}). \quad (47)$$

Notice that polyphase transfer matrix provides the polyphase implementation of the filter bank in Fig. 1, only if the graph satisfies the noble identity condition in (4). Summarizing, we have proved:

Theorem 7 (Polyphase implementation of a filter bank). *On a graph with the adjacency matrix satisfying the noble identity condition (4), the maximally decimated M -channel FB given*

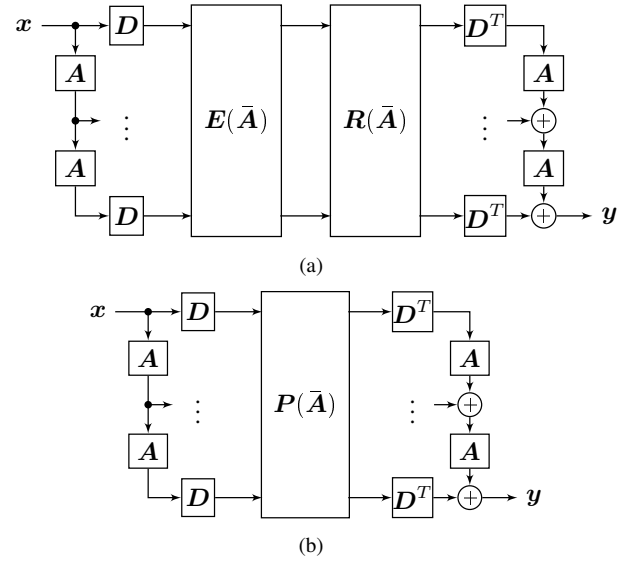


Fig. 4. (a) Polyphase representation of the maximally decimated M -channel filter bank in Fig. 1 on a graph with the adjacency matrix \mathbf{A} that satisfies the noble identity condition (4). (b) Combined representation of the polyphase matrices. The decimation matrix \mathbf{D} is as in (1) and the polyphase transfer matrix is given in (47). $\bar{\mathbf{A}}$ is as in (5).

in Fig. 1 has the polyphase implementation given in Fig. 4(a)-(b) where polyphase transfer matrix, $\mathbf{P}(\bar{\mathbf{A}})$, is as in (47), and $\bar{\mathbf{A}}$ is the adjusted shift operator given in (5). \diamond

Since M -block cyclic matrices satisfy the noble identity condition (4), the polyphase representation of Fig. 4 is valid for graph filter banks on M -block cyclic graphs. However, M -block cyclic property is not *necessary* for this. The condition (4) is enough.

In Sec. III, we showed that it is possible to construct PR filter banks on Ω -graphs. In order to talk about polyphase implementation of such filter banks, we need to characterize the set of graph matrices \mathbf{A} that satisfy both the noble identity condition in (4) and have the Ω -structure in (10). From Theorem 4 and Theorem 5 of [1], we know that M -block cyclic graphs with diagonalizable adjacency matrices belong to that set. It is interesting to observe that any matrix that belongs to this set is *similar* to an M -block cyclic graph. We state this fact in the following theorem whose proof is given in Sec. I of the supplementary document [24].

Theorem 8 (The noble identity condition and the eigenvector structure). *Let \mathbf{A} have distinct eigenvalues with the eigenvalue decomposition $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$. If \mathbf{A} satisfies the noble identity condition in (4) and has the Ω -structure in (10), then \mathbf{A} is similar to an M -block cyclic matrix. More precisely, there exists a permutation matrix $\mathbf{\Pi}$ such that $(\mathbf{V} \mathbf{\Pi}) \mathbf{\Lambda} (\mathbf{V} \mathbf{\Pi})^{-1}$ is M -block cyclic.* \diamond

A. Alias-free and PR Property for M -Block Cyclic Graphs

Returning to Fig. 4 for the polyphase representation of a graph filter bank, we now provide a sufficient condition on $\mathbf{P}(\bar{\mathbf{A}})$ for the PR property on M -block cyclic graphs. This is an extension of a similar condition ($\mathbf{P}(z) = \mathbf{I}$) that guarantees PR in classical filter banks [4].

Theorem 9 (Sufficiency condition for PR in polyphase filter banks on M -block cyclic graphs). *Consider the polyphase implementation of a maximally decimated M -channel filter bank in Fig. 4(b), and assume that the adjacency matrix of the graph, \mathbf{A} , is M -block cyclic. The system has PR property if the polyphase transfer matrix has the following form:*

$$\mathbf{P}(\bar{\mathbf{A}}) = \mathbf{I}_M \otimes \bar{\mathbf{A}}^m, \quad (48)$$

for some non-negative integer m , and $\bar{\mathbf{A}}$ is as in (5). \diamond

Proof: Assume that $\mathbf{P}(\bar{\mathbf{A}})$ has the form in (48). Then, overall response of the filter bank in Fig. 4(b) is written as:

$$\begin{aligned} T(\mathbf{A}) &= \sum_{k=0}^{M-1} \mathbf{A}^{M-1-k} \mathbf{D}^T \bar{\mathbf{A}}^m \mathbf{D} \mathbf{A}^k, \\ &= \sum_{k=0}^{M-1} \mathbf{A}^{M-1-k} \mathbf{D}^T \mathbf{D} \mathbf{A}^{Mm} \mathbf{A}^k, \end{aligned} \quad (49)$$

$$\begin{aligned} &= \left(\sum_{k=0}^{M-1} \mathbf{A}^{M-1-k} \mathbf{D}^T \mathbf{D} \mathbf{A}^k \right) \mathbf{A}^{Mm}, \\ &= \mathbf{A}^{M-1+Mm}, \end{aligned} \quad (50)$$

where we use the first noble identity (2) in (49) and the lazy FB PR property Eq. (31) of [1] in (50), since M -block cyclic matrices satisfy both of these properties. ■

Notice that when we let $m = 0$ in Theorem 9, we get $\mathbf{P}(\bar{\mathbf{A}}) = \mathbf{I}_N$ which corresponds to the lazy filter bank structure given in Fig. 3(b) of [1]. This observation agrees with the result given by Theorem 4 of [1] for M -block cyclic graphs.

For the most complete characterization of the PR property on M -block cyclic graphs, we provide the following result, whose proof is provided in Sec. II of the supplementary document [24].

Theorem 10 (PR polyphase filter banks on M -block cyclic graphs). *Consider the polyphase implementation of a maximally decimated M -channel filter bank in Fig. 4(b), and assume that the adjacency matrix of the graph, \mathbf{A} , is an invertible M -block cyclic matrix. The system has PR property if and only if the polyphase transfer matrix has the following form:*

$$\mathbf{P}(\bar{\mathbf{A}}) = \left(\mathbf{I}_M \otimes \bar{\mathbf{A}}^m \right) \begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-n} \otimes \mathbf{I}_{N/M} \\ \mathbf{I}_n \otimes \bar{\mathbf{A}} & \mathbf{0} \end{bmatrix}, \quad (51)$$

for some integers m, n with $0 \leq n \leq M-1$. $\bar{\mathbf{A}}$ is as in (5). \diamond

When we waive the perfect reconstruction property and ask for an alias-free response, we get a more relaxed condition on the polyphase transfer matrix. The following theorem states the necessary condition for this case. This is a generalization of the classical pseudocirculant condition for alias cancellation [4] to graph filter banks.

Theorem 11 (Alias-free polyphase filter banks on M -block cyclic graphs). *Consider the polyphase implementation of a maximally decimated M -channel filter bank in Fig. 4(b), and assume that the adjacency matrix of the graph, \mathbf{A} , is M -block cyclic with distinct eigenvalues. The system has alias-free property if and only if the polyphase transfer matrix has*

the following pseudo-block-circulant form:

$$\mathbf{P}(\bar{\mathbf{A}}) = \begin{bmatrix} P_0(\bar{\mathbf{A}}) & P_1(\bar{\mathbf{A}}) & \cdots & P_{M-1}(\bar{\mathbf{A}}) \\ \bar{\mathbf{A}} P_{M-1}(\bar{\mathbf{A}}) & P_0(\bar{\mathbf{A}}) & \cdots & P_{M-2}(\bar{\mathbf{A}}) \\ \vdots & \ddots & \ddots & \vdots \\ \bar{\mathbf{A}} P_1(\bar{\mathbf{A}}) & \bar{\mathbf{A}} P_2(\bar{\mathbf{A}}) & \cdots & P_0(\bar{\mathbf{A}}) \end{bmatrix}. \quad (52)$$

\diamond

Proof: Assume that the overall response of the filter bank is alias-free, and the graph has distinct eigenvalues. Then, due to Theorem 11 of [1], the response of the filter bank is a polynomial filter. By re-indexing the coefficients, we can decompose the polynomial response as follows:

$$T(\mathbf{A}) = \sum_{m=0}^{N/M-1} \sum_{k=0}^{M-1} \alpha_{m,k} \mathbf{A}^{M-1+Mm+k}. \quad (53)$$

Since $T(\mathbf{A})$ is a linear combination of PR systems, using Theorem 10 and linearity, we can write the polyphase transfer matrix that corresponds to (53) in the following way:

$$\mathbf{P}(\bar{\mathbf{A}}) = \sum_{m=0}^{N/M-1} \sum_{k=0}^{M-1} \alpha_{m,k} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-k} \otimes \bar{\mathbf{A}}^m \\ \mathbf{I}_k \otimes \bar{\mathbf{A}}^{m+1} & \mathbf{0} \end{bmatrix}}_{\text{pseudo-block-circulant}}. \quad (54)$$

By inspection, we can see that the block-matrix in (54) is a pseudo-block-circulant matrix. Since a linear combination of pseudo-block-circulant matrices is also a pseudo-block-circulant, the polyphase transfer matrix has the form in (52) for some polynomials.

Conversely, if the matrix $\mathbf{P}(\bar{\mathbf{A}})$ has the form in (52), then we can decompose it as in (54) for some set of $\alpha_{m,k}$. Therefore, due to Theorem 10, the overall response of the filter bank is a linear combination of perfect reconstruction systems. It is therefore alias-free. ■

B. Importance of Polyphase Representations for Graph Filter Banks

We know that the polyphase implementation in Fig. 4 is valid as long as the conditions (4) and (5) for noble identities are satisfied. Here each polyphase component $E_{k,l}(\bar{\mathbf{A}}) \in \mathcal{M}^{N/M}$ is a polynomial in the matrix $\bar{\mathbf{A}}$:

$$E_{k,l}(\bar{\mathbf{A}}) = e_{k,l}(0) \mathbf{I} + e_{k,l}(1) \bar{\mathbf{A}} + \cdots + e_{k,l}(K) \bar{\mathbf{A}}^K, \quad (55)$$

where $\bar{\mathbf{A}} = \mathbf{D} \mathbf{A}^M \mathbf{D}^T$ as in (5), and $K \approx L/M$. (L is the order of the filters in Fig 1.) Assuming that $\bar{\mathbf{A}}$ is sparse and can be implemented with negligible overhead, the complexity for $E_{k,l}(\bar{\mathbf{A}})$ (implemented similar to Fig. 7 of [1]) is about $(L/M)(N/M)$ so that the entire polyphase matrix with M^2 such submatrices has complexity LN , identical to the non polyphase implementation of polynomial filter banks (with each filter implemented as in Fig. 7 of [1]). In fact even if \mathbf{A} is sparse, the matrix $\bar{\mathbf{A}} = \mathbf{D} \mathbf{A}^M \mathbf{D}^T$ is in general not. So the polyphase implementation may even have higher complexity than direct implementation of polynomial filters as in Fig. 7 of [1]. Thus, while in classical filter banks the polyphase implementation reduces the number of multiplications per unit time, there is no such advantage in graph filter banks.

Then the question is, what is the advantage of the polyphase representation of Fig. 4 for graph filter banks? The answer lies in the *design phase* rather than the complexity of implementation. To explain, suppose we want to optimize the polynomial analysis filters to achieve certain properties of the decimated subband signals $\{x_0, x_1, \dots\}$ (e.g., sparsity), by using some apriori information on the statistics of the graph signal x . If we do this directly by optimizing the multipliers $h_k(n)$ in the structure of Fig. 7 of [1] then it is not easy to constraint these coefficients (during optimization) such that there will exist a perfect reconstruction polynomial synthesis filter bank $\{F_k(\mathbf{A})\}$. But if we optimize the coefficients $e_{k,l}(n)$ of the polynomials (55), then it can be shown that as long as the equivalent classical polynomial matrix

$$\mathbf{E}(z) = \begin{bmatrix} E_{0,0}(z) & E_{0,1}(z) & \dots & E_{0,M-1}(z) \\ E_{1,0}(z) & E_{1,1}(z) & \dots & E_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & \dots & E_{M-1,M-1}(z) \end{bmatrix} \quad (56)$$

has a polynomial inverse $\mathbf{R}(z)$ (i.e., FIR inverse), there will exist a polynomial graph synthesis bank $\{F_k(\mathbf{A})\}$ with perfect reconstruction property. Now, the construction of classical polynomial matrices $\mathbf{E}(z)$ with polynomial inverses $\mathbf{R}(z)$ is a well studied problem in filter bank theory and includes special families such as FIR paraunitary matrices, FIR unimodular matrices and so forth [4]. So, we can take advantage of the results from classical literature to design optimal graph filter banks that suit specific signal statistics, if we use the polyphase representation of Fig. 4 in the design process. The implementation of each filter in the bank could later be done directly using Fig. 7 of [1], which is more economical.

VI. FREQUENCY DOMAIN ANALYSIS

In order to quantitatively interpret the amount of fluctuation of a signal, the *total variation* of the signal x on a graph with the adjacency matrix \mathbf{A} is defined as [10]:

$$S(x) = \|x - \mathbf{A}x\|_1, \quad (57)$$

where it is assumed that the adjacency matrix is normalized such that the maximum eigenvalue has a unit magnitude [10]. With the definition in (57), the *frequency* of an eigenvector v with the corresponding eigenvalue λ becomes

$$S(v) = |1 - \lambda|, \quad (58)$$

where all the eigenvectors are assumed to be scaled such that they have unit ℓ_1 norm.

When the eigenvalues are *real*, from (58) it is clear that two eigenvectors with distinct eigenvalues cannot have the same amount of total variation. Hence, distinct eigenvalues imply distinct total variations, and vice versa. However, for complex eigenvalues this implication does not hold. As a simple example consider two distinct complex eigenvalues $\lambda_1 = (\sqrt{2}-1)/\sqrt{2}$ and $\lambda_2 = 1/2 + j/2$. Clearly $\lambda_1 \neq \lambda_2$, yet we have $S(v_1) = S(v_2)$.

The problem with complex eigenvalues arises when we want to describe the frequency domain behavior of a polynomial filter, $H(\lambda)$. How the filter suppresses or amplifies

eigenvectors according to their total variation determines the characteristics. When there are complex eigenvalues, a filter may respond differently to eigenvectors with the same total variation. More precisely, we may have $|H(\lambda_1)| \neq |H(\lambda_2)|$ even when $S(v_1) = S(v_2)$. As a result, we may not even be able to define the behavior of the filter (low-pass, high-pass etc.). Therefore, filters cannot be designed independent of the graph spectrum when the spectrum has complex values.

The question we ask here is as follows: under what conditions can we design filters with meaningful behavior *for all graphs* that satisfy such conditions? One obvious answer is the case of real eigenvalues since $S(v_1) = S(v_2)$ implies $\lambda_1 = \lambda_2$, hence $|H(\lambda_1)| = |H(\lambda_2)|$. As a result, a filter behaves in the same way for all graphs with real eigenvalues.

For the complex case, we cannot answer the question in the general sense for the time being. Nonetheless, when we constrain the eigenvalues to be on the unit circle, we have the following result.

Theorem 12 (Magnitude response of a polynomial filter, and unit-modulus eigenvalues). *Let $H(\lambda)$ be a polynomial filter with real coefficients. Then $H(\lambda)$ has a well-defined magnitude response w.r.t. the total variation for all graphs with diagonalizable adjacency matrices with unit modulus eigenvalues.* \diamond

Proof: Let eigenvalues of the adjacency matrix be on the unit circle. Then we have $\lambda = e^{-j\theta}$ for some $0 \leq \theta < 2\pi$. Then, the total variation in (58) reduces to

$$S(v) = |1 - e^{-j\theta}| = 2 |\sin(\theta/2)|. \quad (59)$$

Due to symmetry of (59), eigenvectors will have the same total variation *if and only if* the corresponding eigenvalues are conjugate pairs. Assume that $H(\lambda)$ is a polynomial filter with real coefficients. Then, its magnitude response will have the conjugate symmetry, which is to say $|H(\lambda)| = |H(\lambda^*)|$, for all $|\lambda| = 1$. As a result, even though the same total variation may correspond to a conjugate eigenvalue pair, those eigenvalues will be mapped to the same magnitude. Hence, magnitude response of the filter with respect to the total variation will be well-defined and remain the same for all graphs with unit modulus eigenvalues. \blacksquare

A drawback of (59) is the non-linearity of it in terms of the phase angle. We will demonstrate this in the following example. Consider the filter bank coefficients provided in Table 6.5.1 on page 318 of [4]. These analysis filters are optimized for perfect reconstruction in 3-channel filter banks in the classical multirate theory with synthesis filters having the following coefficients $f_{k,l} = h_{k,14-l}$ [4]. Furthermore, they are designed such that each filter allows only one uniform sub-band to pass. That is, $H_k(z)$ passes frequencies in the range $|\omega| \in [\pi k/3, \pi(k+1)/3]$. Their magnitude responses are shown in Fig. 5(a). When all the eigenvalues of \mathbf{A} are on the unit circle, we can quantify the magnitude response of each analysis filter w.r.t. the total variation due to Theorem 12. However, the pass bands of $H_0(\mathbf{A})$, $H_1(\mathbf{A})$ and $H_2(\mathbf{A})$ in the total variation are $[0, 1]$, $[1, \sqrt{3}]$ and $[\sqrt{3}, 2]$, respectively. They are visualized in Fig. 5(b).

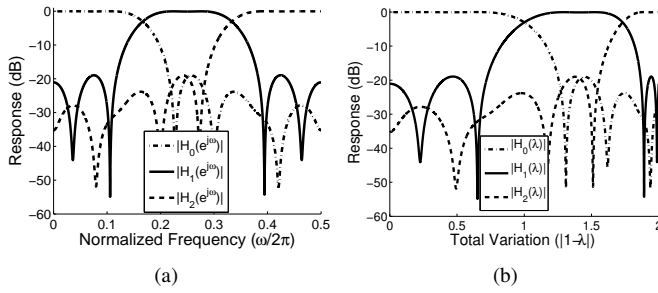


Fig. 5. Magnitude response of the filters given in Table 6.5.1 of [4]. (a) is the magnitude response in the classical theory. (b) is the magnitude response w.r.t. the total variation in (58).

It is important to notice that magnitude responses of the graph filters shown in Fig. 5(b) do *not* explicitly depend on the eigenvalues of the graph as long as they are on the unit circle. *Even repeated eigenvalues are tolerated.* Furthermore, magnitude response and the length of the filters are unrelated with the size of the graph. This makes the system robust to ambiguities in the graph. Remember that for a given specific graph with distinct eigenvalues, we can always construct polynomial filters with the desired magnitude response using Eq. (71) of [1]. However, those filters are unique to that specific graph and quite sensitive to imperfections in the eigenvalues.

Notice that for a maximally decimated filter bank as in Fig. 1, having the PR property and having a well-defined magnitude response are two different concepts, and they do not imply each other. To see this, consider the polynomials in (42). When the adjacency matrix is diagonalizable M -block cyclic, those polynomials provide PR on the filter bank. Yet, they may have a different frequency behavior w.r.t. the total variation for different M -block cyclic graphs. Conversely, we may select the filters with real coefficients. Then, for graphs with unit modulus eigenvalues, we have a graph independent characteristics due to Theorem 12. Yet, we cannot expect them to provide the PR property. Therefore, we reach the following conclusion: even if we assume unit-magnitude eigenvalues in the adjacency matrix, filters given in Table 6.5.1 of [4] do not provide perfect reconstruction on the filter bank. When we further assume that the graph is 3-block cyclic, only then 3-channel filter bank on the graph provides perfect reconstruction with each channel allowing only a sub-band of the total variation spectrum. Moreover, the characteristics of the graph filter bank, PR and magnitude response, will be independent of the actual values of the eigenvalues and the size of the graph, provided that they satisfy the conditions.

The above results show that we can have robust filter banks with practical use for signals on M -block cyclic graphs with eigenvalues having unit magnitude. More importantly, design of such multirate graph systems is not an issue. Due to Theorem 6 and Theorem 12, any algorithm developed for classical signal processing will serve the purpose. We summarize this observation in the following theorem.

Theorem 13 (PR graph filter banks with well-defined magnitude response). *Consider the graph filter bank of Fig. 1, and assume that the adjacency matrix of the graph is diag-*

onalizable M -block cyclic with unit magnitude eigenvalues. If the analysis and the synthesis filters satisfy (29) with real polynomial coefficients, then, the filter bank provides perfect reconstruction, and each channel has a well-defined magnitude response w.r.t. total variation spectra. \diamond

As an application of Theorem 13, consider the cyclic graph C_N . Due to Fact 4 of [1], C_N is an M -block cyclic graph. Furthermore, its eigenvalues are in the form of $\lambda_k = e^{-j2\pi k/N}$ for $0 \leq k \leq N-1$, hence $|\lambda_k| = 1$. As a result, Theorem 13 applies to C_N . Remember that, when the adjacency matrix is C_N , graph signal processing reduces to the classical theory [9]. This observation shows that Theorem 13 agrees with the classical multirate theory and generalizes it to the graph case with some restrictions on the graph.

VII. REMOVING THE NEED FOR Ω -STRUCTURE

In [1], we started with the canonical definition of the decimator as in (1) and extended the classical multirate signal processing theory to graphs. Even though we were able to generalize a number of concepts from classical theory to graph signals, many of the results require \mathbf{A} to be an Ω -graph. In this section we will show that this requirement can be relaxed completely under the mild assumption that \mathbf{A} be diagonalizable. The basic idea is to work with a *similarity-transformed* graph matrix $\tilde{\mathbf{A}} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{-1}$ where \mathbf{Q} is chosen such that $\tilde{\mathbf{A}}$ has the Ω -structure as in (10). We will see that this is always possible. However such a transformation changes the underlying Fourier basis. Therefore, the matrix \mathbf{Q} should be selected carefully, as we will do throughout this section. Given the original graph signal \mathbf{x} we will then define a modified signal

$$\tilde{\mathbf{x}} = \mathbf{Q}\mathbf{x} \quad (60)$$

and use the modified filter bank $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ with canonical decimator (1) to process it (see Fig. 6(a)). Then the filter bank output $\tilde{\mathbf{y}}$ is transformed back to $\mathbf{y} = \mathbf{Q}^{-1}\tilde{\mathbf{y}}$.

The filter bank $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ sandwiched between \mathbf{Q} and \mathbf{Q}^{-1} operates on the modified graph $\tilde{\mathbf{A}}$, which satisfies the eigenvector structure (10). Also, it uses the standard canonical decimator. So, many of the results developed in earlier sections are applicable. We will show that the complete system from \mathbf{x} to \mathbf{y} in Fig. 6(a) is equivalent to the graph filter bank system shown in Fig. 6(b), which operates on the original graph. Notice carefully that the canonical decimator and expander have been replaced in this system by a new decimator and expander (to be defined below in (63)).

The most important point is that the new filter bank $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ that transforms $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{y}}$ and the original filter bank that transforms \mathbf{x} to \mathbf{y} have the following close relationship, as we shall show:

- 1) The filters $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ are polynomials in $\tilde{\mathbf{A}}$ if and only if the original filters $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$ are polynomials in \mathbf{A} .
- 2) The filter bank $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ is a perfect reconstruction system on the graph $\tilde{\mathbf{A}}$ if and only if the original filter bank $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$ is a perfect reconstruction system on the graph \mathbf{A} .

- 3) Assuming that the diagonalizable graph \mathbf{A} has distinct eigenvalues, the filter bank $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ is alias-free on the graph $\tilde{\mathbf{A}}$ if and only if the original filter bank $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$ is alias-free on the graph \mathbf{A} .
- 4) With the input-output map of the filter bank with generalized decimator and expander denoted as $T_G(\mathbf{A})$ and the transformed map as $T(\tilde{\mathbf{A}})$, they are related as in Fig. 6(c)-6(d), that is,

$$T_G(\mathbf{A}) = \mathbf{Q}^{-1} T(\tilde{\mathbf{A}}) \mathbf{Q}. \quad (61)$$

Notice that relations in 1, 2, and 3 follow from the following identity, which holds true for *any* invertible \mathbf{Q} and for *any* polynomial $H(\cdot)$ (Theorem 6.2.9 of [25]):

$$H(\mathbf{A}) = \mathbf{Q}^{-1} H(\tilde{\mathbf{A}}) \mathbf{Q}. \quad (62)$$

Here $\tilde{\mathbf{A}} = \mathbf{Q} \mathbf{A} \mathbf{Q}^{-1}$.

The relation in 4 follows from the following theorem.

Theorem 14 (Generalized filter banks). *Let \mathbf{A} be a diagonalizable adjacency matrix with the eigenvalue decomposition $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$. Let \mathbf{E} be an invertible matrix with the Ω -structure (10) on its columns. Set the similarity transform as $\mathbf{Q} = \mathbf{E} \mathbf{V}^{-1}$. Hence $\tilde{\mathbf{A}} = \mathbf{Q} \mathbf{A} \mathbf{Q}^{-1}$. Define the generalized decimator $\tilde{\mathbf{D}}$ and the generalized expander $\tilde{\mathbf{U}}$ as*

$$\tilde{\mathbf{D}} = \mathbf{D} \mathbf{Q} = \mathbf{D} \mathbf{E} \mathbf{V}^{-1}, \quad \tilde{\mathbf{U}} = \mathbf{Q}^{-1} \mathbf{D}^T = \mathbf{V} \mathbf{E}^{-1} \mathbf{D}^T. \quad (63)$$

Then, a FB $\{H_k(\tilde{\mathbf{A}}), F_k(\tilde{\mathbf{A}})\}$ on $\tilde{\mathbf{A}}$ that uses the canonical decimator and expander, Fig. 6(a), is equivalent to the generalized FB $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$ on \mathbf{A} that uses generalized decimator and expander, Fig. 6(b). Here the term “equivalent” means that the input-output behaviors are related as in (61). \diamond

Proof: Let $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$ be the generalized FB on \mathbf{A} that uses the generalized decimator and expander. Then

$$T_G(\mathbf{A}) = \sum_{k=0}^{M-1} F_k(\mathbf{A}) \tilde{\mathbf{U}} \tilde{\mathbf{D}} H_k(\mathbf{A}), \quad (64)$$

$$= \sum_{k=0}^{M-1} \mathbf{Q}^{-1} \mathbf{Q} F_k(\mathbf{A}) \mathbf{Q}^{-1} \mathbf{D}^T \mathbf{D} \mathbf{Q} H_k(\mathbf{A}) \mathbf{Q}^{-1} \mathbf{Q}, \quad (65)$$

$$= \mathbf{Q}^{-1} \sum_{k=0}^{M-1} F_k(\tilde{\mathbf{A}}) \mathbf{D}^T \mathbf{D} H_k(\tilde{\mathbf{A}}) \mathbf{Q}, \quad (66)$$

$$= \mathbf{Q}^{-1} T(\tilde{\mathbf{A}}) \mathbf{Q}, \quad (67)$$

where we use (62) in (66). Notice that the generalized FB implicitly operates on $\tilde{\mathbf{A}}$, which is an Ω -graph since $\tilde{\mathbf{A}} = \mathbf{Q} \mathbf{A} \mathbf{Q}^{-1} = \mathbf{E} \mathbf{\Lambda} \mathbf{E}^{-1}$. Hence, the eigenvector condition (10) on \mathbf{A} is implicitly satisfied on the generalized FB. \blacksquare

The identity in (62) basically says that instead of working on the given adjacency matrix, we can use the similarity-transformed adjacency matrix as long as the input graph signal is also transformed accordingly. As an example, consider the generalized FB. The matrix \mathbf{Q} transforms the graph signal \mathbf{x} into $\tilde{\mathbf{x}}$ as shown in Fig. 6(a). Special cases of this can be found in [9] where a permutation matrix is used and in [7] where a diagonal matrix is used. Here we use it for a different purpose, namely to create a hypothetical system (the system

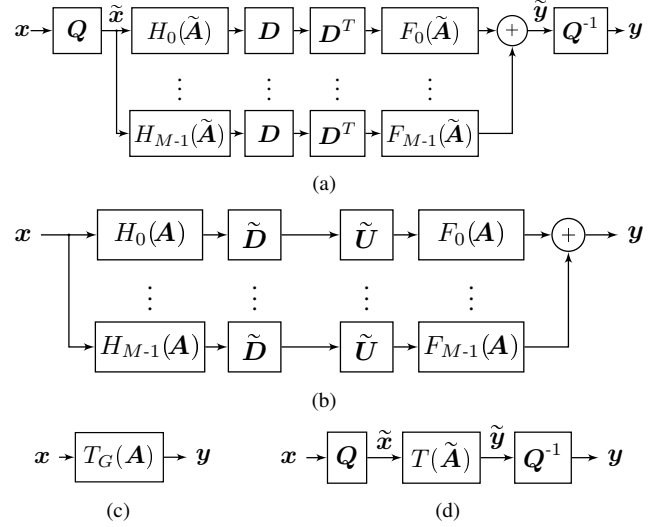


Fig. 6. (a) A FB on the similarity-transformed adjacency matrix $\tilde{\mathbf{A}}$ sandwiched between \mathbf{Q} and \mathbf{Q}^{-1} . (b) The FB with generalized decimator and expander on the original adjacency matrix \mathbf{A} . (c) and (d) are input-output equivalent of the systems in terms of \mathbf{A} and $\tilde{\mathbf{A}}$, respectively. \mathbf{Q} is the similarity transform. \mathbf{D} is as in (1). $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{U}}$ are as in (63). All four systems shown above are equivalent to each other in terms of input-output relations.

flanked by \mathbf{Q} and \mathbf{Q}^{-1} in Fig. 6(a)) that satisfies the eigenvector condition (10). This Ω -structure is sufficient (though possibly not necessary) to be able to use some of the filter banks we developed, e.g., the brickwall filter bank of Theorem 4, and alias free filter banks of Sec. II. Thus the similarity transform \mathbf{Q} merely sets the stage for that. As long as the similarity transform \mathbf{Q} is selected properly, \mathbf{A} can be treated as if it is an Ω -graph even if it is not. For example, consider the spectrum folding phenomena described in Sec. VII of [1]. When the decimator and the expander are selected as in (63), we can remove the condition on the eigenvectors of the adjacency matrix. We state this result as follows.

Theorem 15 (Spectrum folding and generalized decimation). *Let \mathbf{A} be the adjacency matrix of a graph with the following eigenvalue decomposition $\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1}$. Let \mathbf{E} be an invertible matrix with the Ω -structure (10) on its columns. Define the generalized decimator and expander as in (63). Let \mathbf{x} be a signal on the graph \mathbf{A} and \mathbf{y} be the DU version of \mathbf{x} , that is, $\mathbf{y} = \tilde{\mathbf{U}} \tilde{\mathbf{D}} \mathbf{x}$. Then, graph Fourier transform of \mathbf{x} and \mathbf{y} are related as*

$$\hat{\mathbf{y}} = \frac{1}{M} \left(\mathbf{I}_{N/M} \otimes \mathbf{1}_M \mathbf{1}_M^T \right) \hat{\mathbf{x}}, \quad (68)$$

which is nothing but the spectrum folding phenomena. \diamond

Proof: The graph Fourier transforms of \mathbf{x} and \mathbf{y} are given as $\hat{\mathbf{x}} = \mathbf{V}^{-1} \mathbf{x}$ and $\hat{\mathbf{y}} = \mathbf{V}^{-1} \mathbf{y}$, respectively. Therefore we have

$$\mathbf{V} \hat{\mathbf{y}} = \tilde{\mathbf{U}} \tilde{\mathbf{D}} \mathbf{V} \hat{\mathbf{x}}, \quad (69)$$

that is,

$$\begin{aligned} \hat{\mathbf{y}} &= \mathbf{V}^{-1} \mathbf{V} \mathbf{E}^{-1} \mathbf{D}^T \mathbf{D} \mathbf{E} \mathbf{V}^{-1} \mathbf{V} \hat{\mathbf{x}}, \\ &= \mathbf{E}^{-1} \mathbf{D}^T \mathbf{D} \mathbf{E} \hat{\mathbf{x}} \\ &= \frac{1}{M} \left(\mathbf{I}_{N/M} \otimes \mathbf{1}_M \mathbf{1}_M^T \right) \hat{\mathbf{x}}, \end{aligned} \quad (70)$$

where the last equation follows from Eq. (60)-(66) of [1] since E has the Ω -structure. ■

Even though results of Theorem 8 of [1] and Theorem 15 appear to be similar, the difference is that Theorem 8 of [1] applies to diagonalizable Ω -graphs, whereas Theorem 15 applies to *any* graph with diagonalizable adjacency matrix. Notice that Theorem 15 includes Theorem 8 of [1]: when the eigenvectors of A have the Ω -structure, simply select $E = V$, the decimator and the expander then reduce to canonical form $\tilde{D} = D$ and $\tilde{U} = D^T$.

The application of a similarity transform $Q A Q^{-1}$ is therefore useful whenever the Ω -structure is necessary. For arbitrary eigenvectors, we need to use a similarity matrix Q to adjust them to the form (10), which, in turn, affects the choice of decimation matrix: namely the non-canonical form (63) that depends on Q has to be used. On the other hand, this technique does not alter the eigenvalues of A . Remember from Eq. (71) of [1] that coefficients of a polynomial filter and its frequency response are related through the eigenvalues of the graph. Therefore, whatever Q we choose, a polynomial filter on A , $H(A)$, and the same polynomial filter on \tilde{A} , $H(\tilde{A})$, have the same frequency response in their corresponding graph Fourier domains. In summary, for the desired multirate graph signal processing, the coefficients of polynomial filters should be designed according to the graph spectrum (eigenvalues of A); the decimator and the expander should be designed according to eigenvectors of the graph.

The generalized decimator and expander in (63) have two important properties. Firstly, given a graph, they are not unique: we can select E arbitrarily as long as its columns have the Ω -property, and it is invertible. In fact, E can be selected as a properly permuted version of the inverse DFT matrix of size N . This follows from the fact that C_N is an M -block cyclic matrix for any M that divides N . Secondly, the generalized decimator *does* depend on the eigenspaces of the adjacency matrix due to presence of V in (63). Therefore, we cannot talk about a “universal” decimator, and expander, which can remove the eigenvector constraint in (10) for all graphs.

It might appear that the use of Q and Q^{-1} involves additional complexity of the order of N^2 in Fig. 6(a) (where N is the size of the graph). This can be avoided if we implement Fig. 6(b) which is equivalent. In this implementation the simple decimator D is replaced with \tilde{D} . Now, \tilde{D} is an $(N/M) \times N$ matrix with possibly non-zero entries everywhere, and there are M such matrices in the figure. This gives the impression that there is additional computational overhead of N^2 multiplications. But note that \tilde{D} is not unique; it is defined as $\tilde{D} = D E V^{-1}$ where E is an arbitrary matrix with the Ω -structure. The degrees of freedom in E can be exploited to make \tilde{D} relatively sparse to reduce the complexity. In fact \tilde{D} can have the form $\tilde{D} = [I_{N/M} \ X]$ as shown next.

Assume that E has the Ω -structure on its columns. Then it can be written as

$$E = \begin{bmatrix} E_1 (I_{N/M} \otimes f_1^H) \\ \vdots \\ E_M (I_{N/M} \otimes f_M^H) \end{bmatrix}, \quad (71)$$

for some E_k where $E_k \in \mathcal{M}^{N/M}$ and f_k is the k^{th} column of the DFT matrix of size M . Let $R = V^{-1}$ and write it as

$$R = [R_1 \ \cdots \ R_M], \quad (72)$$

where $R_k \in \mathcal{C}^{N \times N/M}$. Then we have the following for the generalized decimator

$$\begin{aligned} \tilde{D} &= D E R = E_1 (I_{N/M} \otimes f_1^H) R, \\ &= [E_1 (I_{N/M} \otimes f_1^H) R_1 \ \cdots \ E_1 (I_{N/M} \otimes f_1^H) R_M]. \end{aligned} \quad (73)$$

Then, select E_1 as follows:

$$E_1 = \left((I_{N/M} \otimes f_1^H) R_1 \right)^{-1}. \quad (74)$$

As a result we get the decimator in the form of $\tilde{D} = [I_{N/M} \ X]$ where X depends on R_k 's. This proves the claim. In this construction the decimator and expander have the form

$$\tilde{D} = E_1 (I_{N/M} \otimes f_1^H) V^{-1}. \quad (75)$$

$$\tilde{U} = V (I_{N/M} \otimes f_1) E_1^{-1}. \quad (76)$$

At the time of this writing, we do not know how to reduce the complexities of both \tilde{D} and \tilde{U} simultaneously.

The similarity transform in (62) changes the eigenvectors, and hence the graph Fourier basis that supports the filter bank. So the transform matrix Q should be selected carefully, otherwise the new Fourier basis may be of no use. Notice that Fig. 6(a) and 6(b) are equivalent, hence the similarity transform can be integrated into the decimator and the expander. At this point, notice the proposed decimator in (75) and the expander in (76). They explicitly *depend* on the Fourier basis of the original graph. As a result, the idea of a similarity transform reduces to a carefully designed decimator, where the decimator in (75) takes the Fourier basis of the original graph into account and decimates the signal accordingly. The same interpretation is valid for the expander in (76) as well. Moreover, the filters in Fig. 6(b) are still polynomials in the original graph, hence the original graph Fourier basis can still be used to diagonalize the filter responses.

VIII. EXAMPLES

In this section, we will implement a 3-channel brickwall filter bank for the famous Minnesota road graph [6], [7], [12]. With due thanks to the authors of [6] and [12], we use the data publicly available in [26], [27]. This graph has 2642 nodes in total where 2 nodes are disconnected to the rest of the graph. Since a road graph is expected to be connected, we disregard those two nodes. See Fig. 7 for the visual representation of the graph. The adjacency matrix is extremely sparse with only 0.1% non-zero entries. Since the edges are represented with 1's, the unit shift, Ax , merely requires addition and no multiplication at all.

It is clear from Fig. 7 that the graph is not 3-block cyclic. Therefore, we cannot apply Theorem 4 directly. However, we can use the generalized decimator and expander as explained in Sec. VII. Furthermore, we select the graph Laplacian to be the unit shift element and find the graph Fourier basis, V , as the eigenvectors of the graph Laplacian. This selection

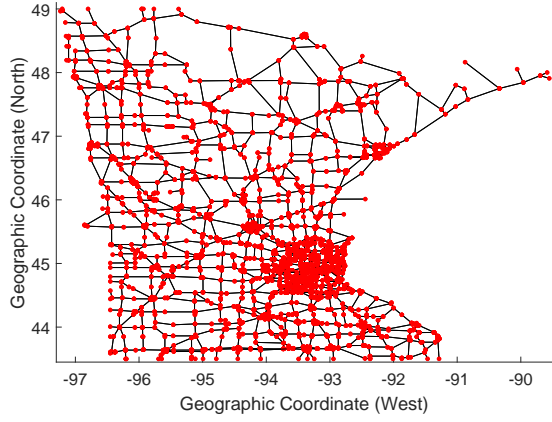


Fig. 7. Minnesota traffic graph which has $N = 2640$ nodes, and 3302 undirected unweighted edges (courtesy of [6], [12]).

(the Laplacian but not the adjacency matrix) is consistent with the development, since we do not put any specific meaning to the unit shift operator. We precisely use (75) and (76) to construct the generalized decimator and expander, respectively. We construct the analysis and synthesis filters as in (26). We denote the reconstructed output of the k^{th} channel as $\mathbf{y}_k = \mathbf{F}_k \tilde{\mathbf{U}} \tilde{\mathbf{D}} \mathbf{H}_k \mathbf{x}$ for $0 \leq k \leq 2$. (These are indicated in Fig. 2(b)).

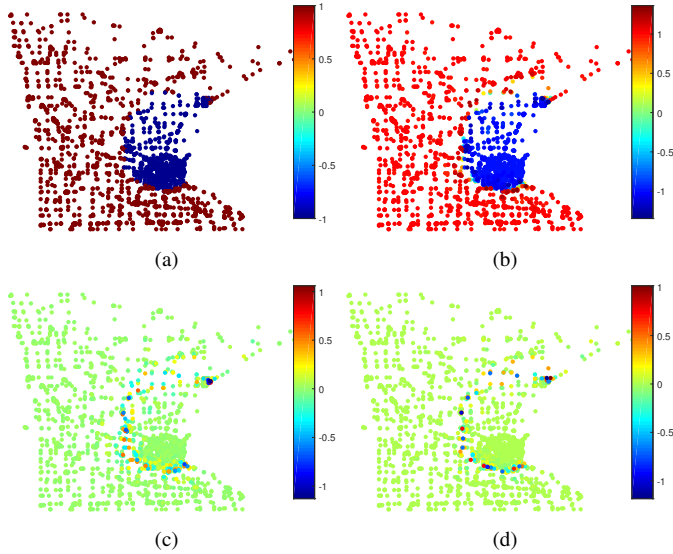


Fig. 8. (a) Signal consisting of only 1's and -1's as given in [7], output of (b) channel-0, (c) channel-1, (d) channel-2.

In the first example, we consider the signal used in [6], [7] as the filter bank input. A visual representation of this signal is given in Fig. 8(a). The reconstructed outputs from subbands, $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2$, are given in Fig. 8(b), 8(c), 8(d), respectively. We observe that \mathbf{y}_0 is a smooth approximation of the original signal since it is the output of the low-pass channel. The remaining two channels give information about nodes where the discontinuity (high-frequency content) in the signal occurs.

In the second example, we select a smoother signal defined as $x_i = \exp(-0.5 d_i^2)$ where d_i denotes the geographic distance between the i^{th} node and the point $[-93.5 \ 45]$. This signal, which is used as the filter bank input, is visualized in Fig. 9(a).

The reconstructed outputs from the channels, $\mathbf{y}_0, \mathbf{y}_1$, and \mathbf{y}_2 , are given in Fig. 9(b), 9(c), 9(d), respectively. Similar to the previous example, the low-pass channel captures most of the energy. Even though the remaining two channels identify the nodes where change in the signal is located, outputs are not as strong as the previous example. This is due to smoother character of the signal.

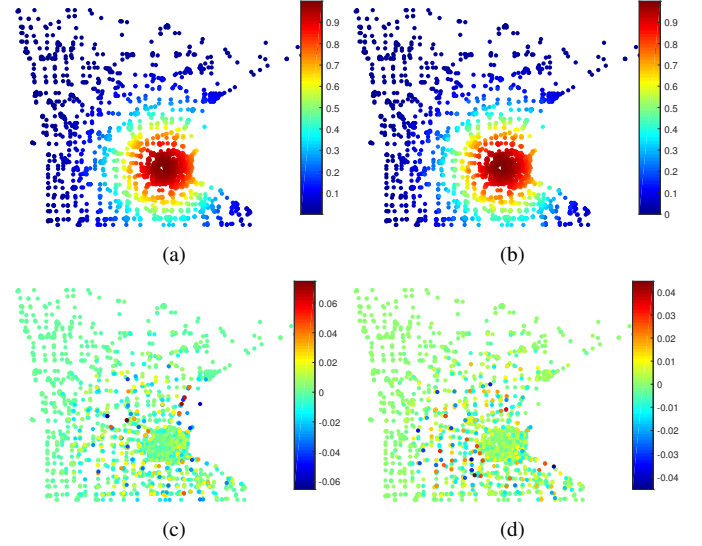


Fig. 9. (a) Signal with exponential decay due to geographic distance, output of (b) channel-0, (c) channel-1, (d) channel-2.

The Laplacian of the Minnesota traffic graph has repeated eigenvalues. Furthermore, the distinct eigenvalues are closely spaced. As a result, we cannot implement $\{\mathbf{H}_k\}$'s as polynomials, as low order polynomial approximations result in poor performance. Hence, each filtering operation requires $N^2 \approx 7 \cdot 10^7$ multiplications. In the following, in order to obtain filters with low complexity, we will replace the brick-wall filters, $\{\mathbf{H}_k\}$, with their hard-thresholded counter-parts, $\{\mathbf{H}'_k\}$, that are constructed as follows:

$$(\mathbf{H}'_k)_{i,j} = \begin{cases} (\mathbf{H}_k)_{i,j}, & |(\mathbf{H}_k)_{i,j}| \geq \gamma \\ 0, & \text{otherwise} \end{cases} \quad (77)$$

for some threshold γ . It is clear that this non-linear operation on the filters compromises the PR property of the filter bank. However, it is interesting to investigate the trade-off between the reconstruction error and the efficiency of the thresholded filters. For this purpose, we define the average fraction of non-zero elements of the filters as $\frac{1}{M N^2} \sum_{k=0}^{M-1} \|\mathbf{H}'_k\|_0$ where $\|\cdot\|_0$ counts the number of non-zeros and define the reconstruction error as $\|\mathbf{x} - \mathbf{x}'\|_2^2 / \|\mathbf{x}\|_2^2$ where \mathbf{x}' is the output of the filter bank with filters in (77). By sweeping over different thresholds, γ , we obtain the result in Fig. 10. Here \mathbf{x}_a denotes the signal in Fig. 8(a), \mathbf{x}_b denotes the signal in Fig. 9(a), and \mathbf{x}_c is a signal with i.i.d. Gaussian entries. These are the inputs to the filter bank in the three cases. It is very interesting to observe that when we tolerate 1% error in the reconstruction, we can sparsify the filters significantly: we only need 7.6%, 12.8% and 3.5% of the non-zero values of the actual filters, respectively. This is a huge saving, due to N^2 being very large. We do not expect hard-thresholding to be optimal, and we are

not suggesting this as a filter design approach. This example merely shows that by replacing the PR property with near-PR property, very efficient filters can be designed for M -channels.

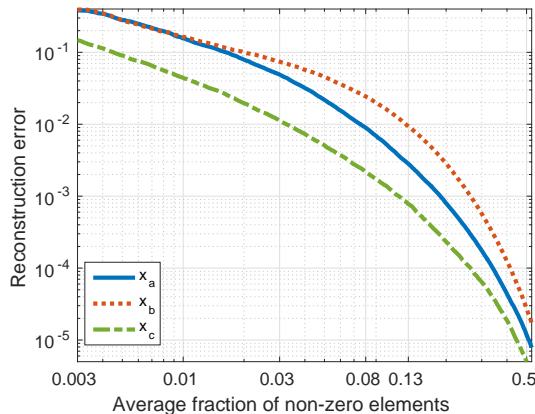


Fig. 10. Trade-off between the efficiency of the filters and the reconstruction error. Trade-off depends on the input signal under consideration.

IX. CONCLUDING REMARKS

In this paper we extended the theory of M -channel maximally decimated filter banks, from classical time domain to the domain of graphs. There are several important aspects in which these filter banks differ from classical filter banks. We found that perfect reconstruction (PR) can be achieved with polynomial filters for graphs which satisfy certain eigenstructure conditions. Furthermore for M -block cyclic graphs, PR filter banks can be built by starting from any classical filter bank. We also developed polyphase representations for such filter banks. Even though many of the results were developed for graphs with a certain eigenstructure, it was shown in Sec. VII that the eigenvector structure (10) can be relaxed, and only the eigenvalue structure (9) remains to be satisfied.

This also brings up some practical questions which we have not been able to address here: what are practical examples of graphs which satisfy the eigenvalue constraints? How do these filter banks perform for practical graph signals, and how do they compare with alternative ways of processing these graph signals? For example, imagine we build a compression system (akin to a subband coder) based on the PR graph filter bank with a certain order for the polynomial filters $\{H_k(\mathbf{A}), F_k(\mathbf{A})\}$. How does this compare with a brute-force compression system that performs a large DFT or a graph Fourier transform on the entire graph signal \mathbf{x} , performs optimal bit allocation, and reconstructs with the inverse transform? Many such practical questions remain to be addressed. In this theoretically intense paper it has not been possible to address these important practical issues, but we plan to explore these aspects in future work.

X. ACKNOWLEDGMENTS

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REFERENCES

- [1] O. Teke and P. P. Vaidyanathan, "Extending classical multirate signal processing theory to graphs – Part I: Fundamentals," *IEEE Trans. Signal Process.*, to appear.
- [2] O. Teke and P. P. Vaidyanathan, "Fundamentals of multirate graph signal processing," in *Asilomar Conf. on Signals, Systems and Computers*, Nov 2015, pp. 1791–1795.
- [3] O. Teke and P. P. Vaidyanathan, "Graph filter banks with m -channels, maximal decimation, and perfect reconstruction," in *Proc. Int. Conf. Acoust. Speech, Signal Process. (ICASSP)*, March 2016, pp. 4089–4093.
- [4] P. P. Vaidyanathan, *Multirate systems and filter banks*. Englewood Cliffs, N.J. Prentice Hall, 1993.
- [5] M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding*. Prentice-Hall, 1995.
- [6] S. Narang and A. Ortega, "Perfect reconstruction two-channel wavelet filter banks for graph structured data," *IEEE Trans. Signal Process.*, vol. 60, no. 6, pp. 2786–2799, June 2012.
- [7] S. Narang and A. Ortega, "Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs," *IEEE Trans. Signal Process.*, vol. 61, no. 19, pp. 4673–4685, Oct 2013.
- [8] D. Shuman, S. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains," *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 83–98, May 2013.
- [9] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs," *IEEE Trans. Signal Process.*, vol. 61, no. 7, pp. 1644–1656, April 2013.
- [10] A. Sandryhaila and J. M. F. Moura, "Discrete signal processing on graphs: Frequency analysis," *IEEE Trans. Signal Process.*, vol. 62, no. 12, pp. 3042–3054, June 2014.
- [11] A. Sandryhaila and J. M. F. Moura, "Big data analysis with signal processing on graphs: Representation and processing of massive data sets with irregular structure," *IEEE Signal Processing Magazine*, vol. 31, no. 5, pp. 80–90, Sept 2014.
- [12] D. K. Hammond, P. Vandergheynst, and R. Gribonval, "Wavelets on graphs via spectral graph theory," *Applied and Computational Harmonic Analysis*, vol. 30, no. 2, pp. 129 – 150, 2011.
- [13] N. Tremblay and P. Borgnat, "Subgraph-based filterbanks for graph signals," *IEEE Trans. Signal Process.*, vol. 64, no. 15, pp. 3827–3840, Aug. 2016.
- [14] V. Ekambaram, G. Fanti, B. Ayazifar, and K. Ramchandran, "Spline-like wavelet filterbanks for multiresolution analysis of graph-structured data," *IEEE Trans. Signal and Information Process. over Networks*, vol. 1, no. 4, pp. 268–278, Dec 2015.
- [15] Y. Tanaka and A. Sakiyama, "M-channel oversampled graph filter banks," *IEEE Trans. Signal Process.*, vol. 62, no. 14, pp. 3578–3590, July 2014.
- [16] S. Chen, R. Varma, A. Sandryhaila, and J. Kovacevic, "Discrete signal processing on graphs: Sampling theory," *IEEE Trans. Signal Process.*, vol. 63, no. 24, pp. 6510–6523, Dec 2015.
- [17] S. Chen, A. Sandryhaila, and J. Kovacevic, "Sampling theory for graph signals," in *Proc. Int. Conf. Acoust. Speech, Signal Process. (ICASSP)*, April 2015, pp. 3392–3396.
- [18] S. Narang, A. Gadde, and A. Ortega, "Signal processing techniques for interpolation in graph structured data," in *Proc. Int. Conf. Acoust. Speech, Signal Process. (ICASSP)*, May 2013, pp. 5445–5449.
- [19] A. Anis, A. Gadde, and A. Ortega, "Towards a sampling theorem for signals on arbitrary graphs," in *Proc. Int. Conf. Acoust. Speech, Signal Process. (ICASSP)*, May 2014, pp. 3864–3868.
- [20] A. Gadde and A. Ortega, "A probabilistic interpretation of sampling theory of graph signals," in *Proc. Int. Conf. Acoust. Speech, Signal Process. (ICASSP)*, April 2015, pp. 3257–3261.
- [21] X. Wang, P. Liu, and Y. Gu, "Local-set-based graph signal reconstruction," *IEEE Trans. Signal Process.*, vol. 63, no. 9, pp. 2432–2444, May 2015.
- [22] A. Marques, S. Segarra, G. Leus, and A. Ribeiro, "Sampling of graph signals with successive local aggregations," *IEEE Trans. Signal Process.*, vol. 64, no. 7, pp. 1832–1843, April 2016.
- [23] P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial," *Proceedings of the IEEE*, vol. 78, no. 1, pp. 56–93, Jan 1990.
- [24] O. Teke and P. P. Vaidyanathan. (2016) Supplementary material for extending classical multirate signal processing theory to graphs – Part II. [Online]. Available: <http://systems.caltech.edu/dsp/students/oteke/files/supp2.pdf>

- [25] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1994.
- [26] S. Narang and A. Ortega. (2013) Graph bior wavelet toolbox. [Online]. Available: http://biron.usc.edu/wiki/index.php/Graph_Filterbanks
- [27] D. K. Hammond, P. Vandergheynst, and R. Gribonval. The spectral graph wavelets toolbox. [Online]. Available: <http://wiki.epfl.ch/sgwt>



P. P. Vaidyanathan (S'80–M'83–SM'88–F'91) was born in Calcutta, India on Oct. 16, 1954. He received the B.Sc. (Hons.) degree in physics and the B.Tech. and M.Tech. degrees in radiophysics and electronics, all from the University of Calcutta, India, in 1974, 1977 and 1979, respectively, and the Ph.D degree in electrical and computer engineering from the University of California at Santa Barbara in 1982. He was a post doctoral fellow at the University of California, Santa Barbara from Sept. 1982 to March 1983. In March 1983 he joined the electrical engineering department of the California Institute of Technology as an Assistant Professor, and since 1993 has been Professor of electrical engineering there.

His main research interests are in digital signal processing, multirate systems, wavelet transforms, signal processing for digital communications, genomic signal processing, radar signal processing, and sparse array signal processing.

Dr. Vaidyanathan served as Vice-Chairman of the Technical Program committee for the 1983 IEEE International symposium on Circuits and Systems, and as the Technical Program Chairman for the 1992 IEEE International symposium on Circuits and Systems. He was an Associate editor for the IEEE Transactions on Circuits and Systems for the period 1985–1987, and is currently an associate editor for the journal IEEE Signal Processing letters, and a consulting editor for the journal Applied and computational harmonic analysis. He has been a guest editor in 1998 for special issues of the IEEE Trans. on Signal Processing and the IEEE Trans. on Circuits and Systems II, on the topics of filter banks, wavelets and subband coders.

Dr. Vaidyanathan has authored nearly 500 papers in journals and conferences, and is the author/coauthor of the four books Multirate systems and filter banks, Prentice Hall, 1993, Linear Prediction Theory, Morgan and Claypool, 2008, and (with Phoong and Lin) Signal Processing and Optimization for Transceiver Systems, Cambridge University Press, 2010, and Filter Bank Transceivers for OFDM and DMT Systems, Cambridge University Press, 2010. He has written several chapters for various signal processing handbooks. He was a recipient of the Award for excellence in teaching at the California Institute of Technology for the years 1983–1984, 1992–93 and 1993–94. He also received the NSF's Presidential Young Investigator award in 1986. In 1989 he received the IEEE ASSP Senior Award for his paper on multirate perfect-reconstruction filter banks. In 1990 he was recipient of the S. K. Mitra Memorial Award from the Institute of Electronics and Telecommunications Engineers, India, for his joint paper in the IETE journal. In 2009 he was chosen to receive the IETE students' journal award for his tutorial paper in the IETE Journal of Education. He was also the coauthor of a paper on linear-phase perfect reconstruction filter banks in the IEEE SP Transactions, for which the first author (Truong Nguyen) received the Young outstanding author award in 1993. Dr. Vaidyanathan was elected Fellow of the IEEE in 1991. He received the 1995 F. E. Terman Award of the American Society for Engineering Education, sponsored by Hewlett Packard Co., for his contributions to engineering education. He has given several plenary talks including at the IEEE ISCAS-04, Sampta-01, Eusipco-98, SPCOM-95, and Asilomar-88 conferences on signal processing. He has been chosen a distinguished lecturer for the IEEE Signal Processing Society for the year 1996–97. In 1999 he was chosen to receive the IEEE CAS Society's Golden Jubilee Medal. He is a recipient of the IEEE Signal Processing Society's Technical Achievement Award for the year 2002, and the IEEE Signal Processing Society's Education Award for the year 2012. He is a recipient of the IEEE Gustav Kirchhoff Award (an IEEE Technical Field Award) in 2016, for "Fundamental contributions to digital signal processing." In 2016 he also received the Northrup Grumman Prize for excellence in Teaching at Caltech.



Oguzhan Teke (S'12) received the B.S. and the M.S. degree in electrical and electronics engineering both from Bilkent University, Turkey in 2012, and 2014, respectively. He is currently pursuing the Ph.D. degree in electrical engineering at the California Institute of Technology, USA.

His research interests include signal processing, graph signal processing, and convex optimization.